Raabe's test

Theorem – Suppose that $u_n > 0$ and that $\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$

Then the series is cgnt. if $k > 1$ and divergent if $k < 1$.

Proof:- Let us compare the given series $\sum u_n$ with the auxiliary series

$$\sum v_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}$$

whose n^{th} term $v_n = \frac{1}{n^p}$. We know that the series $\sum v_n$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

Now $$\frac{v_n}{v_{n+1}} = \frac{1}{n^p} \Big/ \frac{1}{(n+1)^p}$$

$$= \frac{(n+1)^p}{n^p} = \left(\frac{n+1}{n}\right)^p$$

$$= \left(1 + \frac{1}{n}\right)^p$$

$$= 1 + \frac{p}{n} + \frac{p(p-1)}{2n^2} + \text{---}$$

Case I. Suppose that ΣV_n is convergent and hence $p > 1$. Then by the Comparison test ΣU_n is convergent if

$$\frac{U_n}{U_{n+1}} > \frac{V_n}{V_{n+1}}$$

i.e if $\dfrac{U_n}{U_{n+1}} > 1 + \dfrac{p}{n} + O\left(\dfrac{1}{n^2}\right)$

i.e if $\dfrac{U_n}{U_{n+1}} - 1 > \dfrac{p}{n} + O\left(\dfrac{1}{n^2}\right)$

i.e if $n\left(\dfrac{U_n}{U_{n+1}} - 1\right) > p + O\left(\dfrac{1}{n}\right)$

i.e if $Lt_n \left(\dfrac{U_n}{U_{n+1}} - 1\right) >, p \; (>1)$

i.e $\underset{}{Lt}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1$

Case II Suppose that ΣV_n is ~~converg~~ divergent and hence $p < 1$
Then Σu_n is divergent.

if $\frac{u_n}{u_{n+1}} < \frac{V_n}{V_{n+1}}$

i.e. if $Lt\ n\left(\frac{u_n}{u_{n+1}} - 1\right) < p$

i.e. if $Lt\ n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1$

as shown before
Hence the theorem.

Note Raabe's test is applied when Ratio test fails, i.e. when $\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = 1$